

# Balanced Trees

## Part One

# Balanced Trees

- Balanced search trees are among the most useful and versatile data structures.
- Many programming languages ship with a balanced tree library.
  - C++: `std::map` / `std::set`
  - Java: `TreeMap` / `TreeSet`
- Many advanced data structures are layered on top of balanced trees.
  - Curious? Ask me!

# Where We're Going

- ***B-Trees (Today)***
  - A simple type of balanced tree developed for block storage.
- ***Red/Black Trees (Today/Thursday)***
  - The canonical balanced binary search tree.
- ***Augmented Search Trees (Thursday)***
  - Adding extra information to balanced trees to supercharge the data structure.
- ***Randomized BSTs (Next Tuesday)***
  - Do we even need to balance trees?

# Outline for Today

- ***BST Review***

- Refresher on basic BST concepts and runtimes.

- ***Overview of Red/Black Trees***

- What we're building toward.

- ***B-Trees and 2-3-4 Trees***

- Simple balanced trees, in depth.

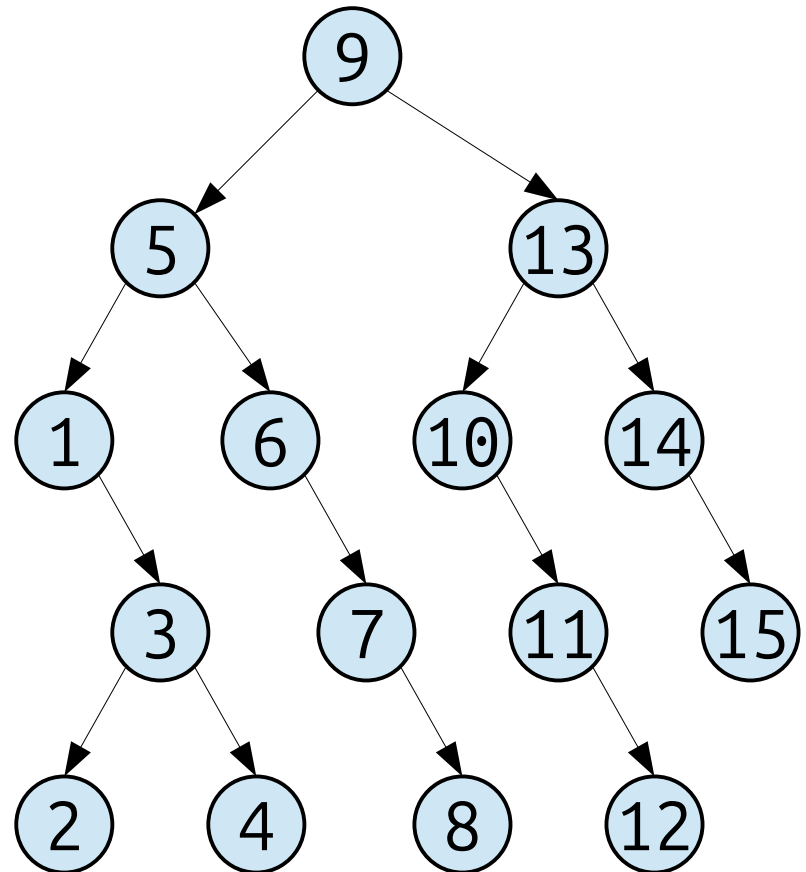
- ***Intuiting Red/Black Trees***

- A much better feel for red/black trees.

# A Quick BST Review

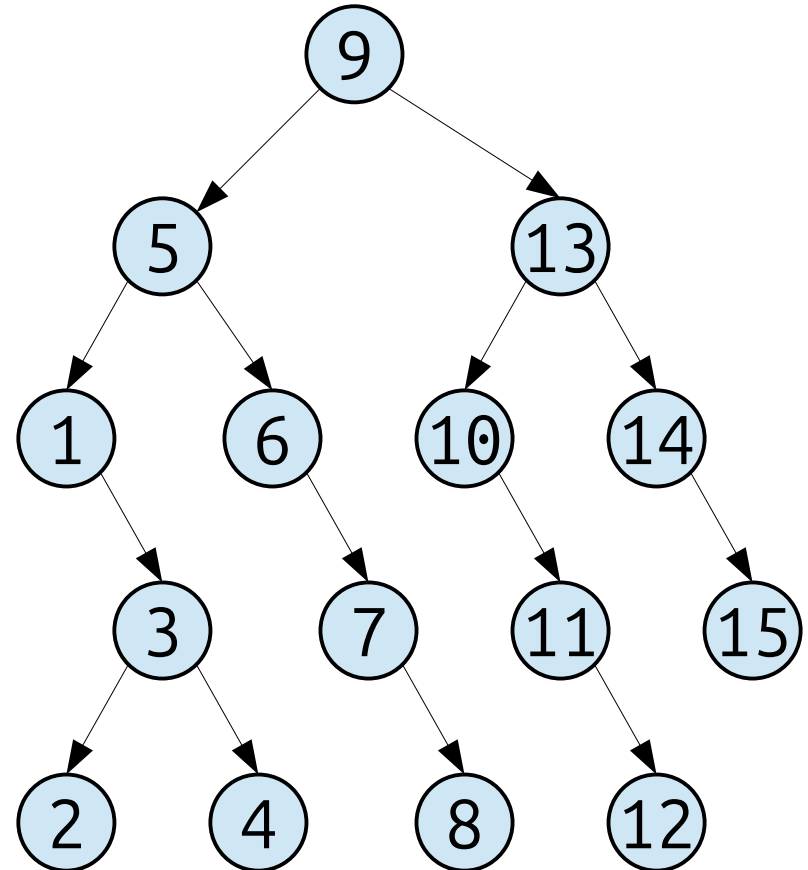
# Binary Search Trees

- A **binary search tree** is a binary tree with the following properties:
  - Each node in the BST stores a **key**, and optionally, some auxiliary information.
  - The key of every node in a BST is strictly greater than all keys to its left and strictly smaller than all keys to its right.
- **Note:** Keys and nodes are related but are not synonymous. You'll see why later.



# Binary Search Trees

- The **height** of a binary search tree is the length of the longest path from the root to a leaf, measured in the number of *edges*.
  - A tree with one node has height 0.
  - A tree with no nodes has height -1, by convention.
- The height of a BST bounds the costs of most basic operations (search, insert, lookup, successor, max, etc.)



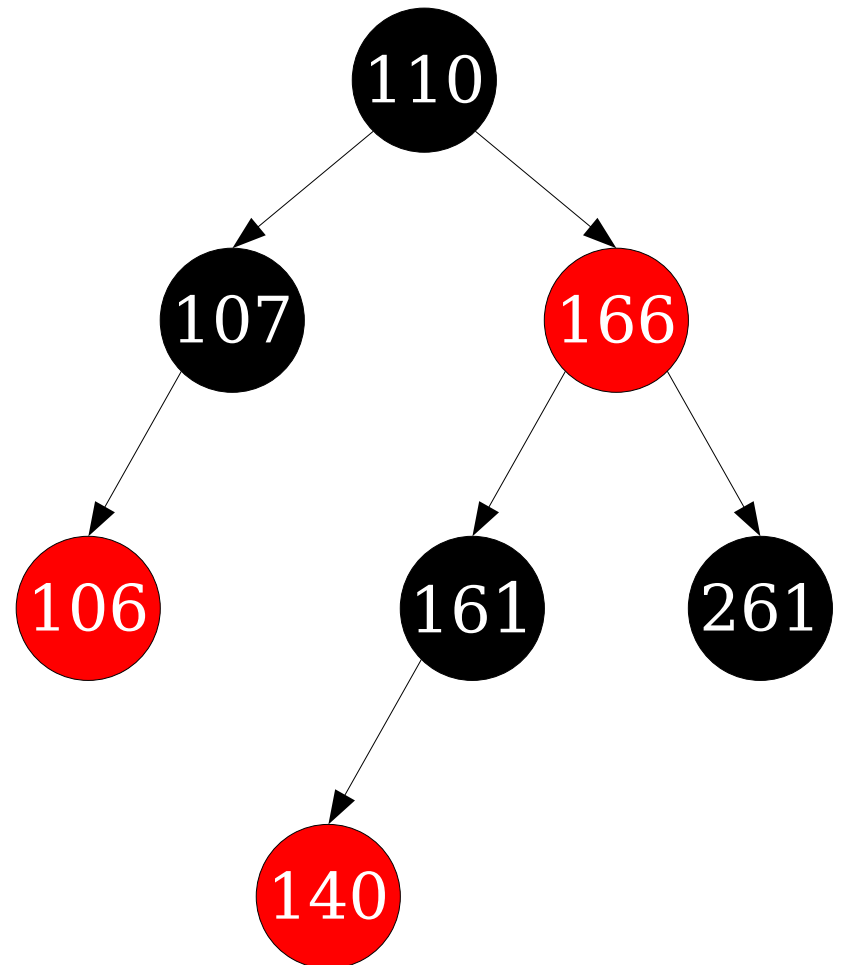
# Runtime Analysis

- The time complexity of all these operations is  $O(h)$ , where  $h$  is the height of the tree.
  - That's the longest path we can take.
- In the best case,  $h = O(\log n)$  and all operations take time  $O(\log n)$ .
- In the worst case,  $h = \Theta(n)$  and some operations will take time  $\Theta(n)$ .
- **Challenge:** How do you efficiently keep the height of a tree low?

# A Glimpse of Red/Black Trees

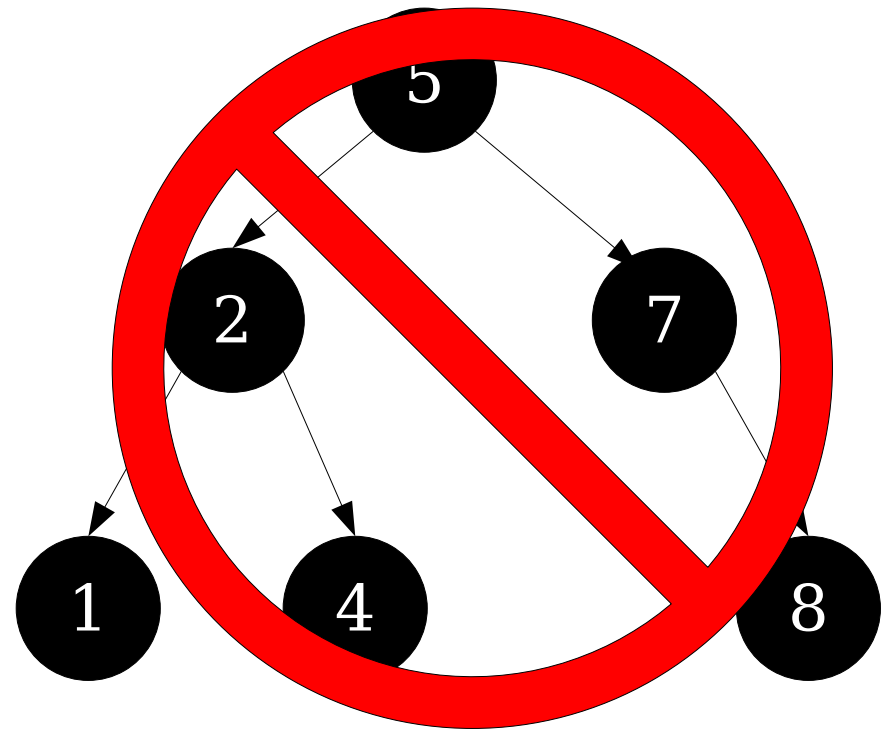
# Red/Black Trees

- A **red/black tree** is a BST with the following properties:
  - Every node is either red or black.
  - The root is black.
  - No red node has a red child.
  - Every root-null path in the tree passes through the same number of black nodes.



# Red/Black Trees

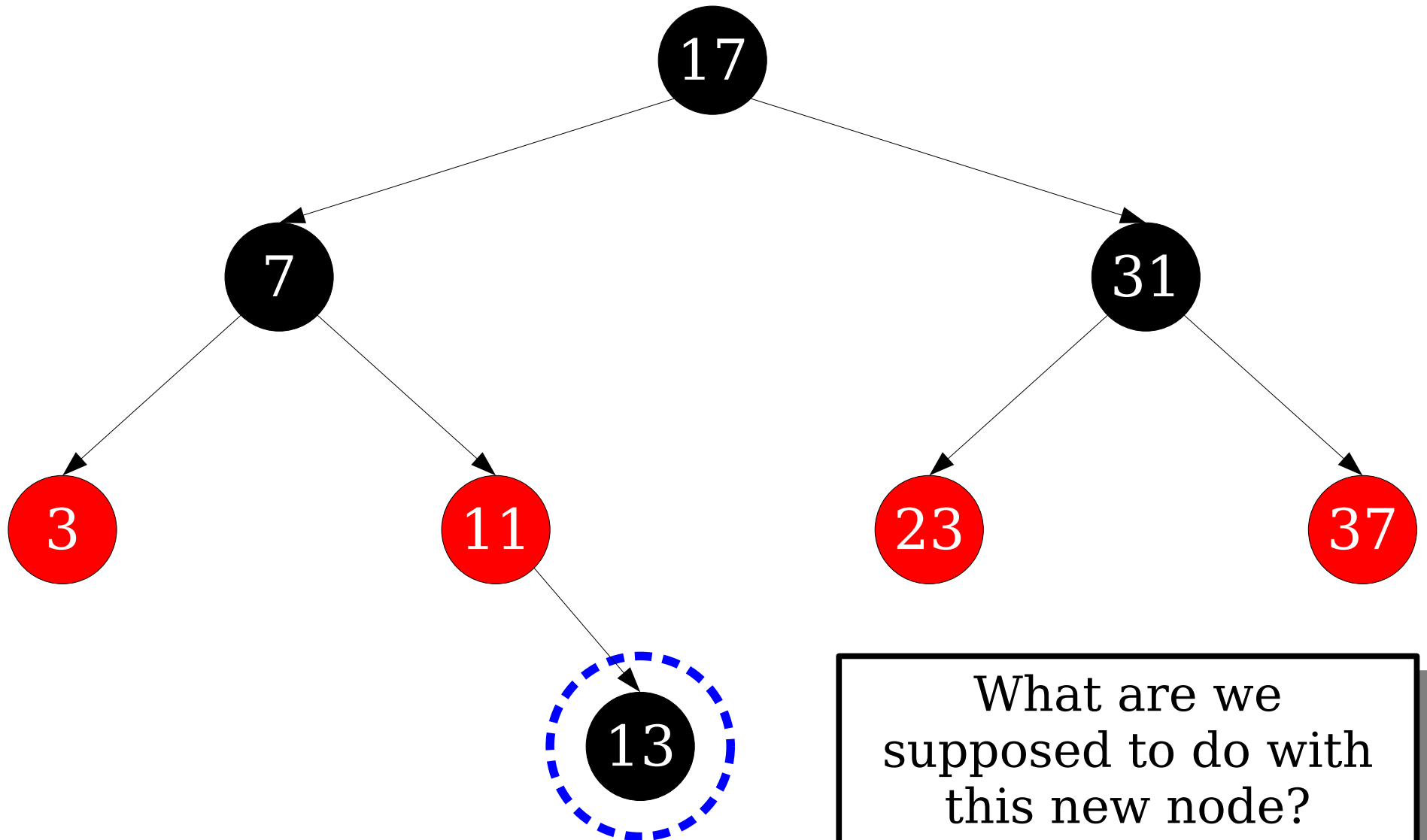
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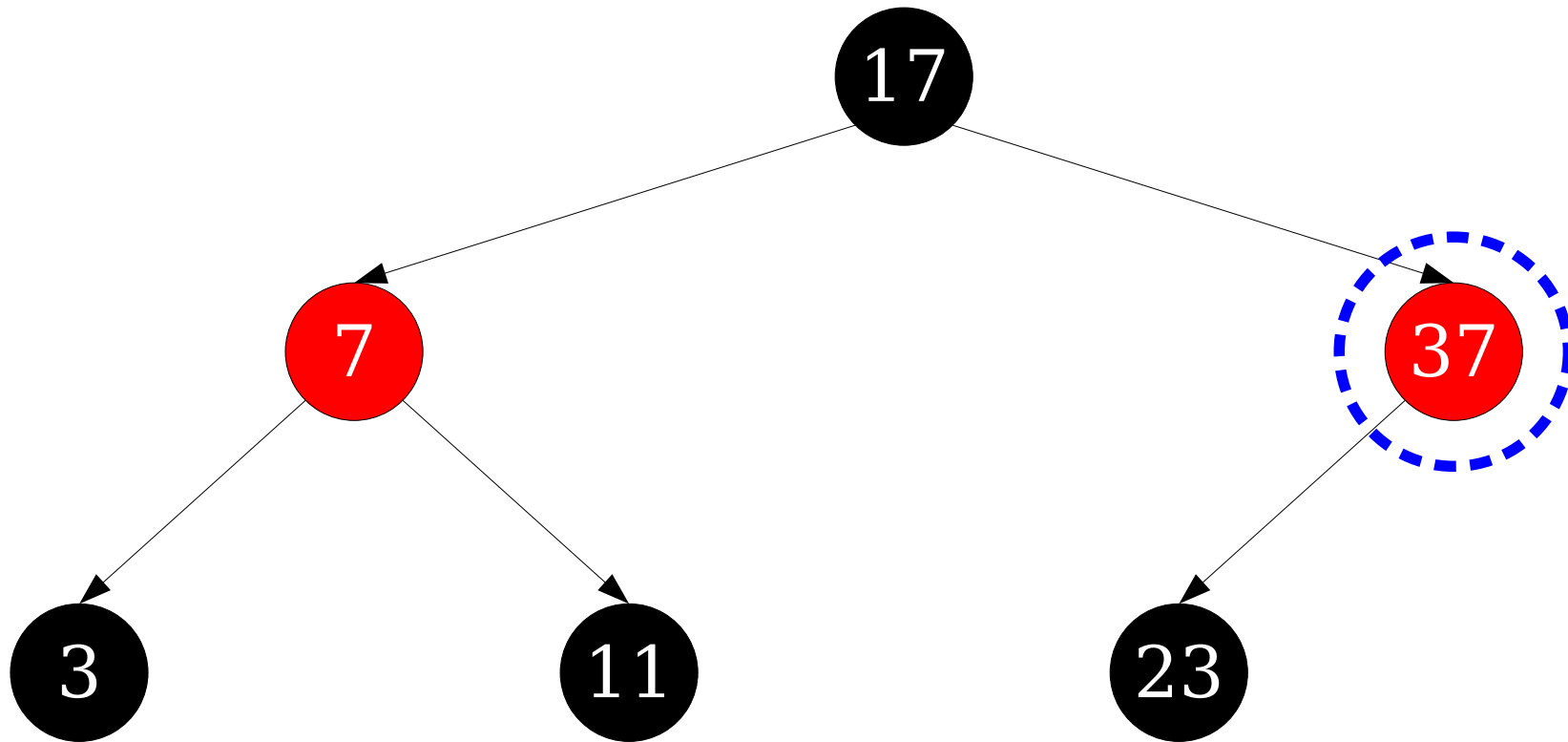
# Red/Black Trees

- ***Theorem:*** Any red/black tree with  $n$  nodes has height  $O(\log n)$ .
  - We could prove this now, but there's a *much* simpler proof of this we'll see later on.
- Given a fixed red/black tree, lookups can be done in time  $O(\log n)$ .

# Mutating Red/Black Trees



# Mutating Red/Black Trees



How do we fix up the black-height property?

# Fixing Up Red/Black Trees

- ***The Good News:*** After doing an insertion or deletion, we can locally modify a red/black tree in time  $O(\log n)$  to fix up the red/black properties.
- ***The Bad News:*** There are a *lot* of cases to consider and they're not trivial.
- Some questions:
  - How do you memorize / remember all the rules for fixing up the tree?
  - How on earth did anyone come up with red/black trees in the first place?

**Time-Out for Announcements!**

# Lecture Participation Opt-Out

- By default, lecture participation (PollEV) accounts for 5% of your course grade.
- If you'd like to opt out of lecture participation and add that extra 5% to your final exam, you can opt out by this Friday at 11:59PM.
- Check Ed for the link you can use to do this.

# Problem Set 2

- Problem Set 1 is graded and solutions are now up on Gradescope.
- Problem Set 2 is due Thursday at 1:00PM.
  - Friendly reminder for the coding component: don't try doing this all in one go. Break it down into smaller, more easily testable pieces.
  - Remember to write beautiful code: decompose complex functions into multiple helpers, comment aggressively, etc.
- Stop by OH or ask on Ed if you have any questions!

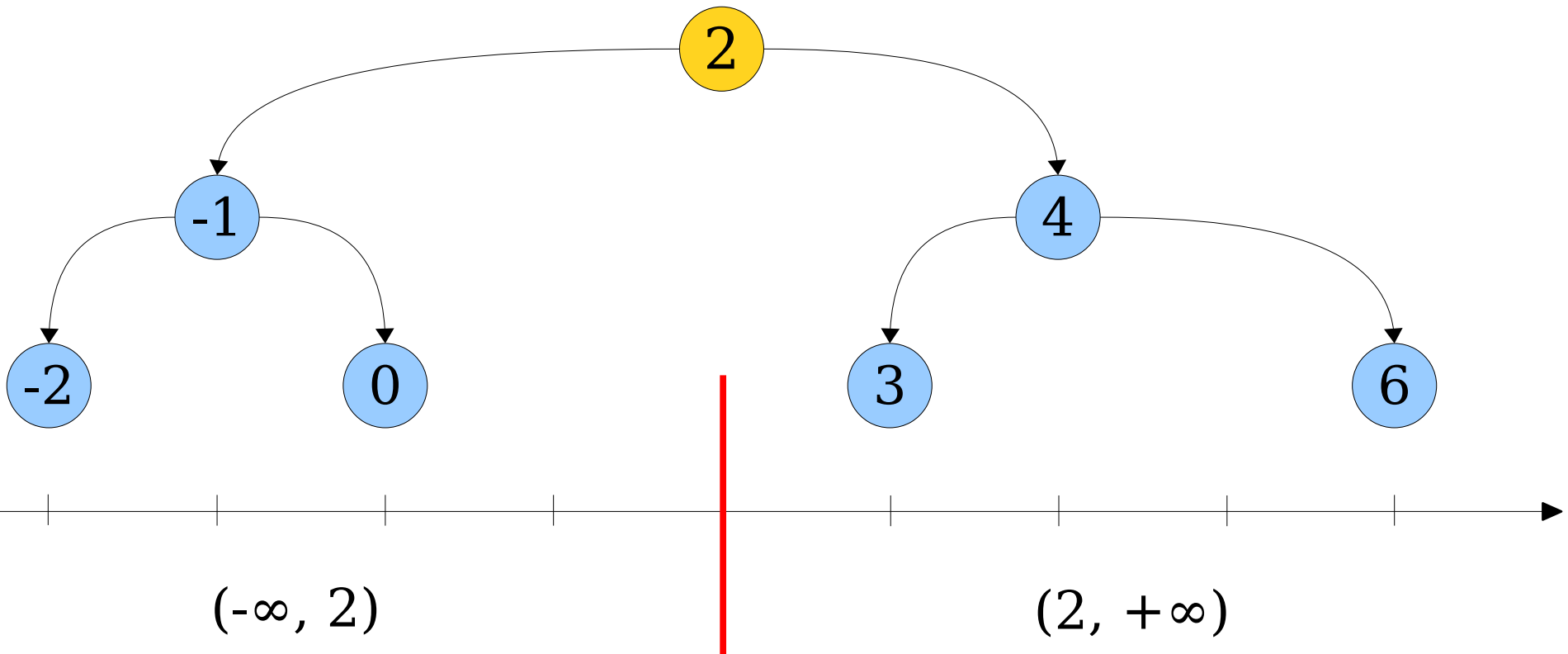
Back to CS166!

How did anyone come up with  
red/black trees in the first place?

# Multiway Search Trees

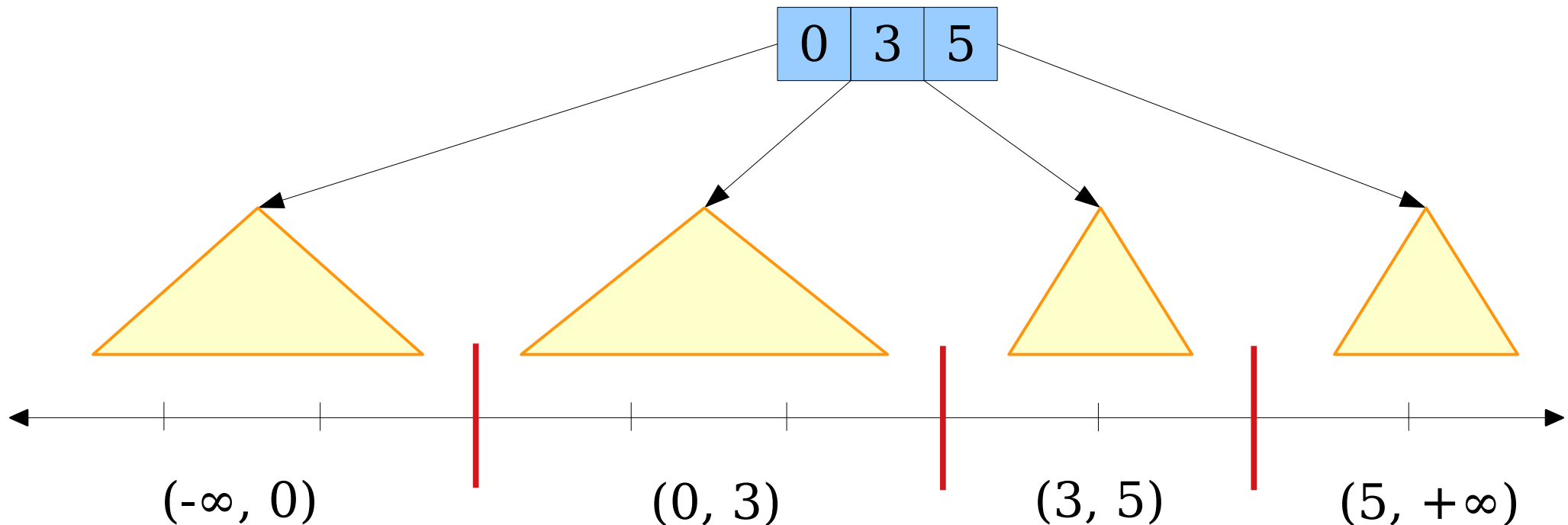
# Generalizing BSTs

- In a binary search tree, each node stores a single key.
- That key splits the “key space” into two pieces, and each subtree stores the keys in those halves.



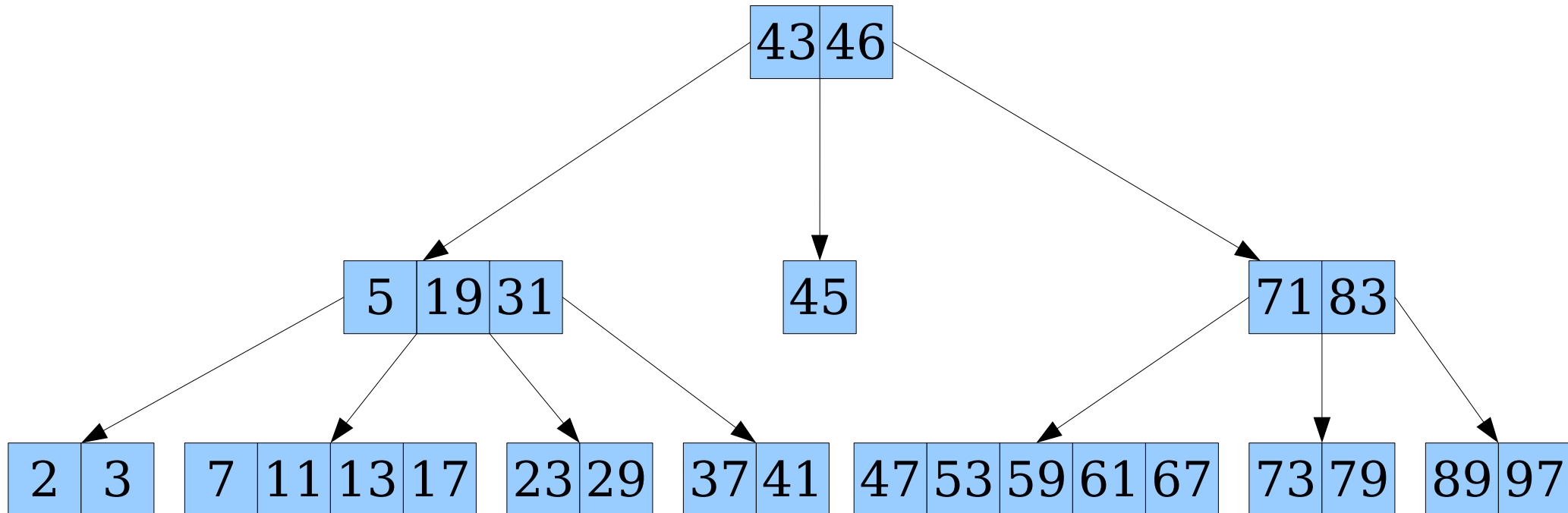
# Generalizing BSTs

- In a **multiway search tree**, each node stores an arbitrary number of keys in sorted order.
- A node with  $k$  keys splits the key space into  $k+1$  regions, with subtrees for keys in each region.



# Generalizing BSTs

- In a *multiway search tree*, each node stores an arbitrary number of keys in sorted order.



- Surprisingly, it's a bit easier to build a balanced multiway tree than it is to build a balanced BST. Let's see how.

# Balanced Multiway Trees

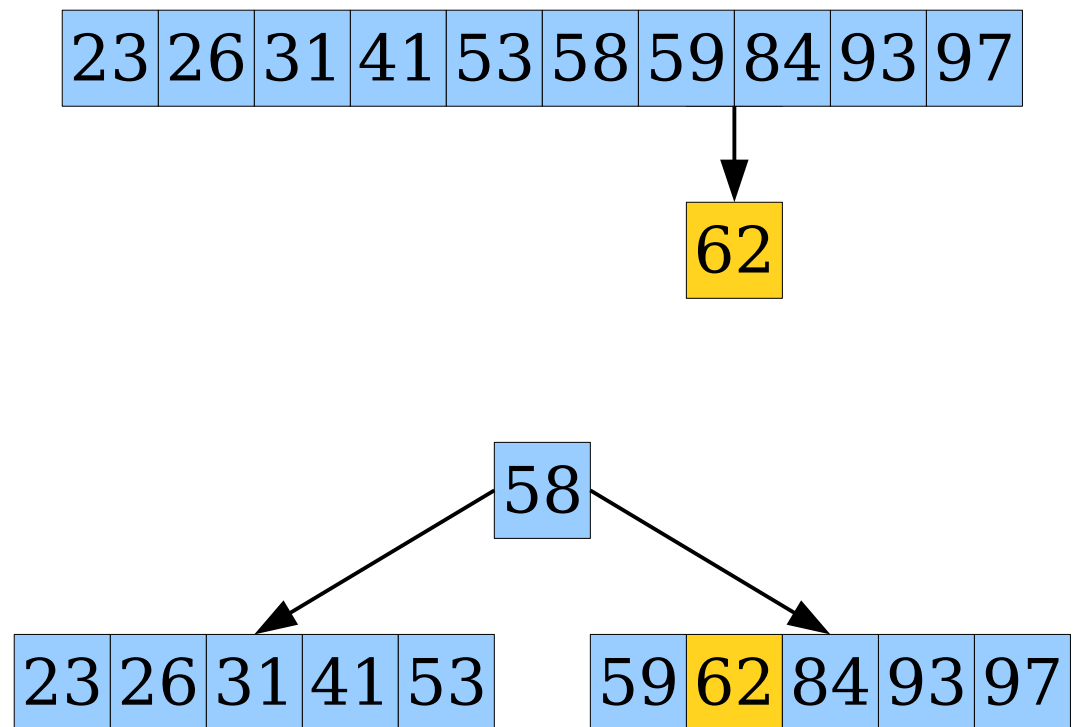
- In some sense, building a balanced multiway tree isn't all that hard.
- We can always just cram more keys into a single node!

23	26	31	41	53	58	59	62	84	93	97
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- At a certain point, this stops being a good idea – it's basically just a sorted array. What does “balance” even mean here?

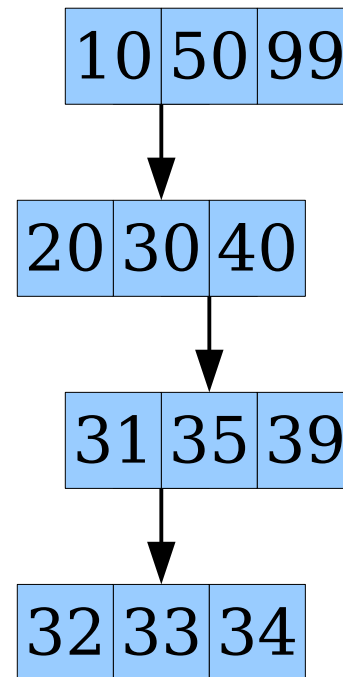
# Balanced Multiway Trees

- What could we do if our nodes get too big?
- **Option 1:** Push the new key down into its own node.
- **Option 2:** Split big nodes in half, kicking the middle key up.
- Assume that we add keys to the deepest node possible.
- Assume we cap the number of keys per node at some value  $k$ .



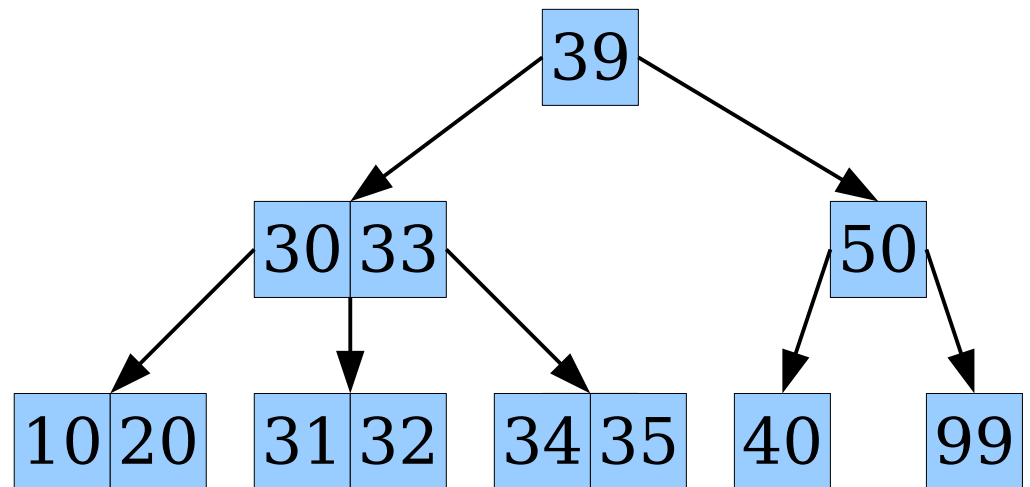
# Balanced Multiway Trees

- **Option 1:** Push keys down into new nodes.
  - Simple to implement.
  - Can lead to tree imbalances.



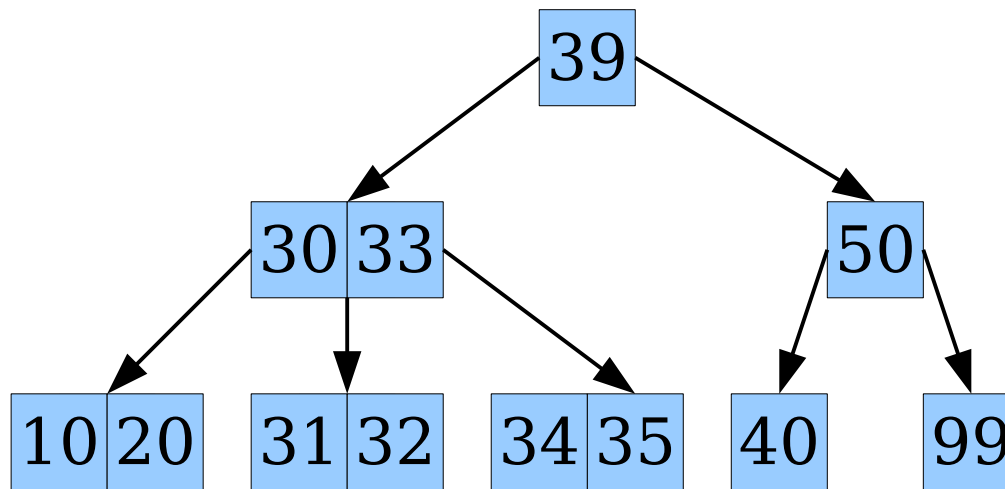
# Balanced Multiway Trees

- **Option 1:** Push keys down into new nodes.
  - Simple to implement.
  - Can lead to tree imbalances.
- **Option 2:** Split big nodes, kicking keys higher up.
  - Keeps the tree balanced.
  - Slightly trickier to implement.



# Balanced Multiway Trees

- **General idea:** Cap the maximum number of keys in a node. Add keys into leaves. Whenever a node gets too big, split it and kick one key higher up the tree.



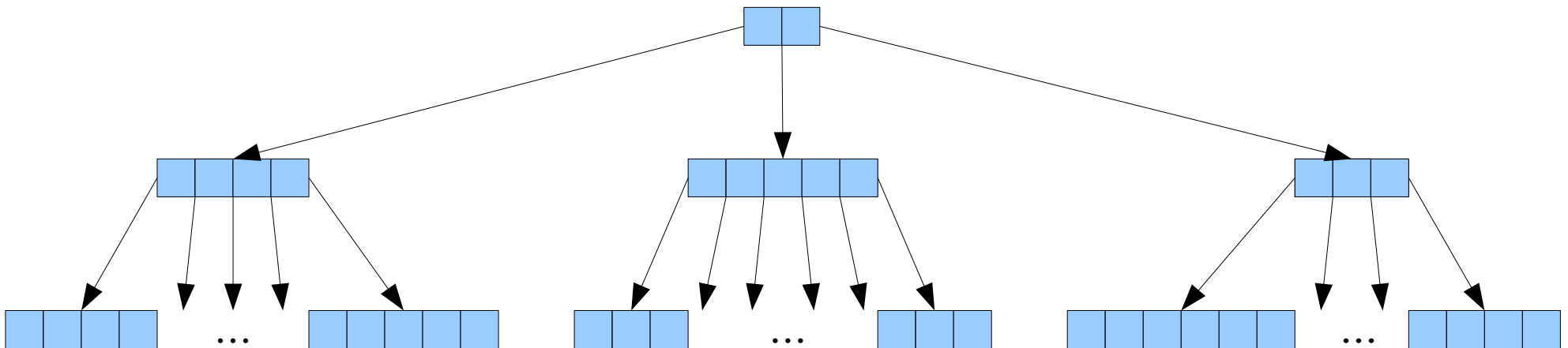
- **Advantage 1:** The tree is always balanced.
- **Advantage 2:** Insertions and lookups are pretty fast.

# Balanced Multiway Trees

- We currently have a ***mechanical description*** of how these balanced multiway trees work:
  - Cap the size of each node.
  - Add keys into leaves.
  - Split nodes when they get too big and propagate the splits upward.
- We currently don't have an ***operational definition*** of how these balanced multiway trees work.
  - e.g. "A Cartesian tree for an array is a binary tree that's a min-heap and whose inorder traversal gives back the original array."

# B-Trees

- A ***B-tree of order  $b$***  is a multiway search tree where
  - each node has between  $b-1$  and  $2b-1$  keys, except the root, which may have between 1 and  $2b-1$  keys;
  - each node is either a leaf or has one more child than key; and
  - all leaves are at the same depth.
- Different authors give different bounds on how many keys can be in each node. The ranges are often  $[b-1, 2b-1]$  or  $[b, 2b]$ . For the purposes of today's lecture, we'll use the range  $[b-1, 2b-1]$  for the key limits, just for simplicity.



# Analyzing B-Trees

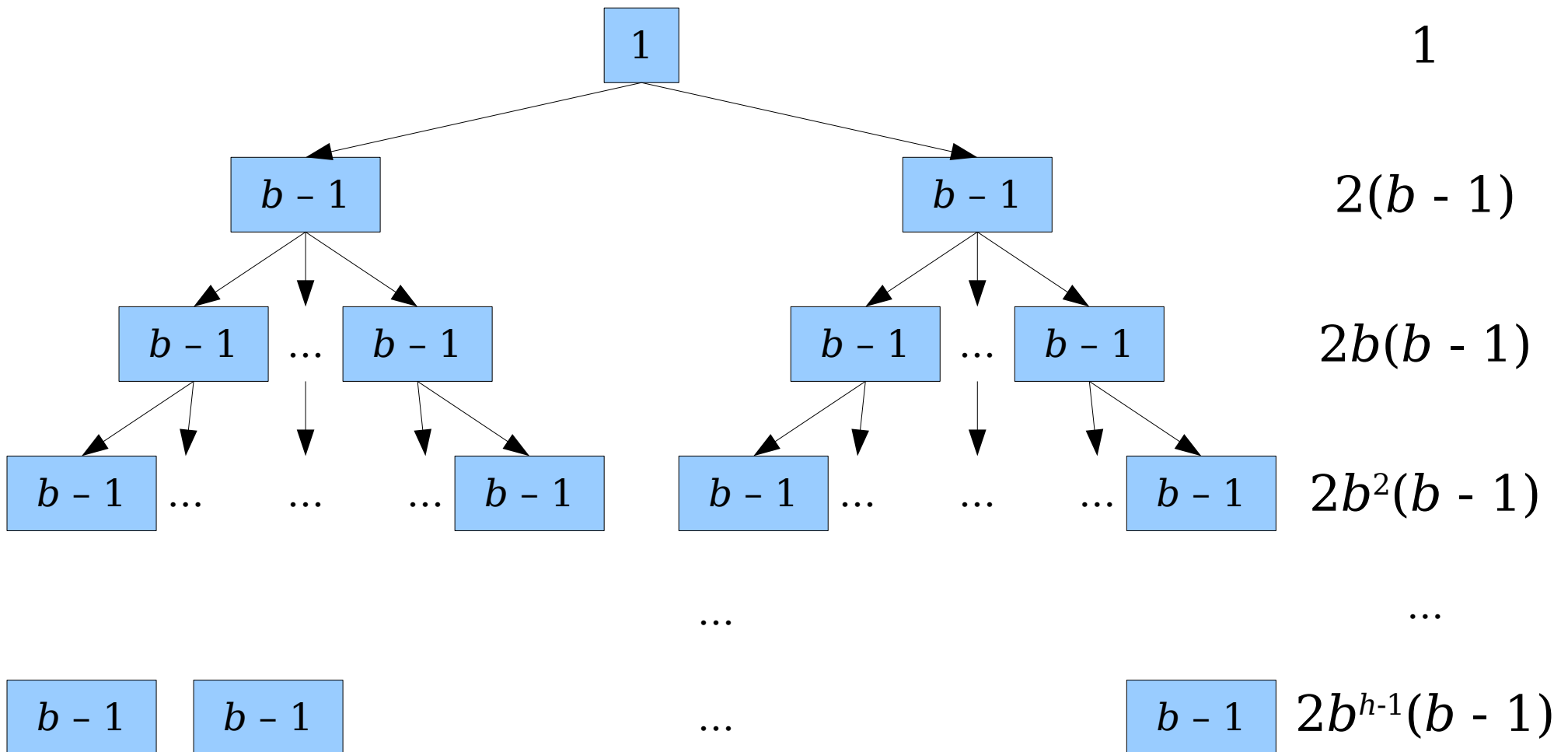
# The Height of a B-Tree

- What is the maximum possible height of a B-tree of order  $b$  that holds  $n$  keys?

***Intuition:*** The branching factor of the tree is at least  $b$ , so the number of keys per level grows exponentially in  $b$ . Therefore, we'd expect something along the lines of  $O(\log_b n)$ .

# The Height of a B-Tree

- What is the maximum possible height of a B-tree of order  $b$  that holds  $n$  keys?



# The Height of a B-Tree

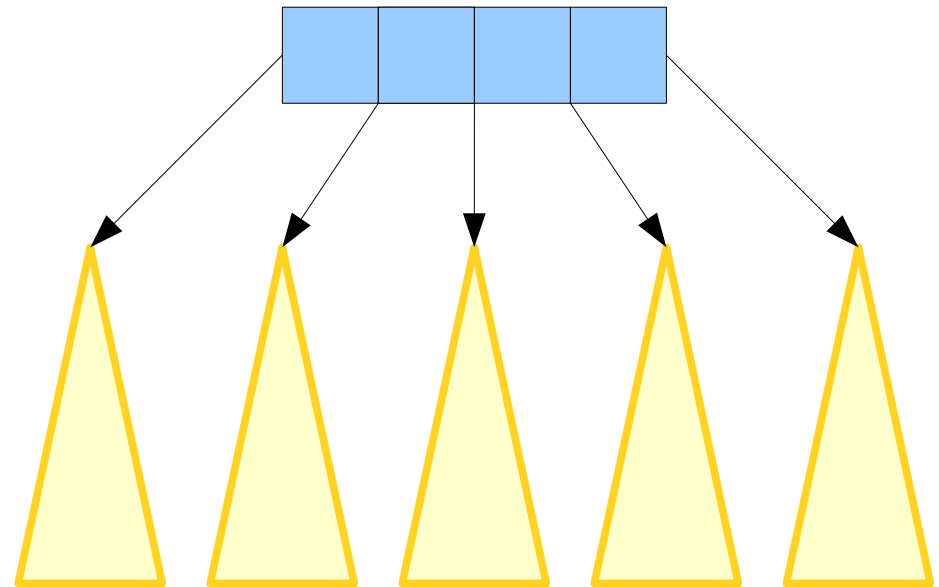
- **Theorem:** The maximum height of a B-tree of order  $b$  containing  $n$  keys is  $O(\log_b n)$ .
- **Proof:** Number of keys  $n$  in a B-tree of height  $h$  is guaranteed to be at least

$$\begin{aligned} & 1 + 2(\mathbf{b} - \mathbf{1}) + 2\mathbf{b}(\mathbf{b} - \mathbf{1}) + 2\mathbf{b}^2(\mathbf{b} - \mathbf{1}) + \dots + 2\mathbf{b}^{h-1}(\mathbf{b} - \mathbf{1}) \\ &= 1 + 2(\mathbf{b} - \mathbf{1})(1 + \mathbf{b} + \mathbf{b}^2 + \dots + \mathbf{b}^{h-1}) \\ &= 1 + 2(\mathbf{b} - \mathbf{1})((\mathbf{b}^h - 1) / (\mathbf{b} - \mathbf{1})) \\ &= 1 + 2(\mathbf{b}^h - 1) = 2\mathbf{b}^h - 1. \end{aligned}$$

Solving  $n = 2b^h - 1$  yields  $h = \log_b ((n + 1) / 2)$ , so the height is  $O(\log_b n)$ . ■

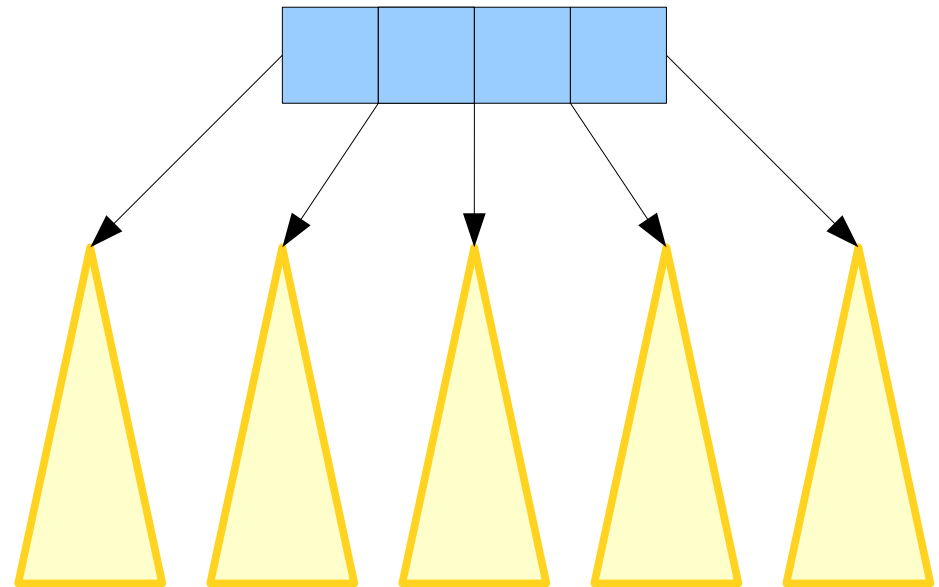
# Analyzing Efficiency

- Suppose we have a B-tree of order  $b$ .
- What is the worst-case runtime of looking up a key in the B-tree?
- **Answer:** It depends on how we do the search!



# Analyzing Efficiency

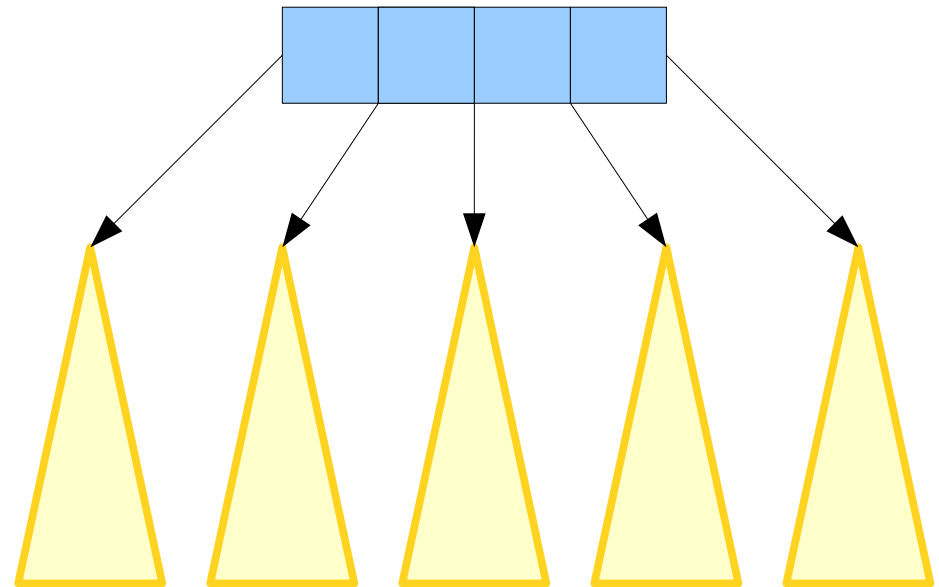
- To do a lookup in a B-tree, we need to determine which child tree to descend into.
- This means we need to compare our query key against the keys in the node.
- **Question:** How should we do this?



# Analyzing Efficiency

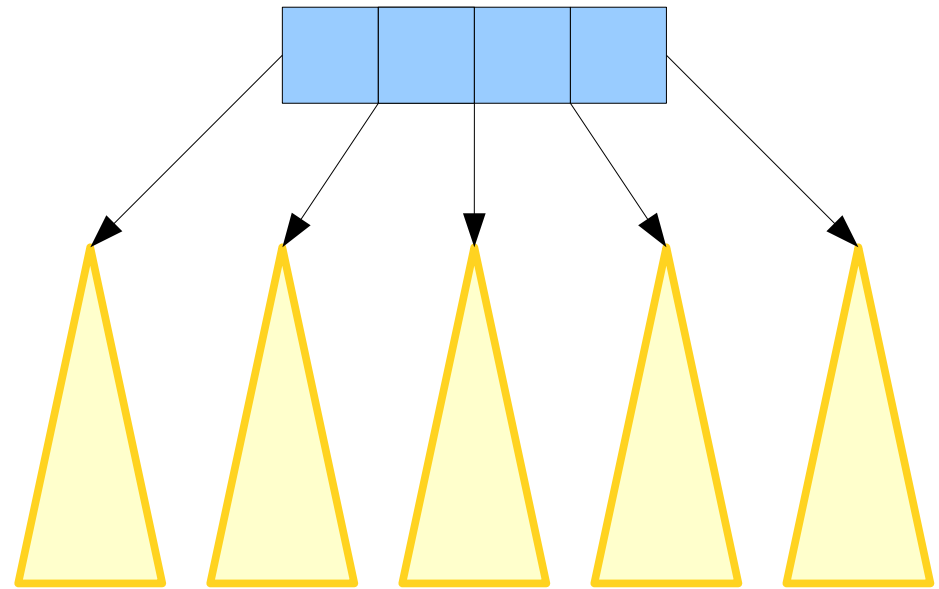
- **Option 1:** Use a linear search.
- Cost per node:  $O(b)$ .
- Nodes visited:  $O(\log_b n)$ .
- Total cost:

$$\begin{aligned} & O(b) \cdot O(\log_b n) \\ &= \mathbf{O(b \log_b n)} \end{aligned}$$



# Analyzing Efficiency

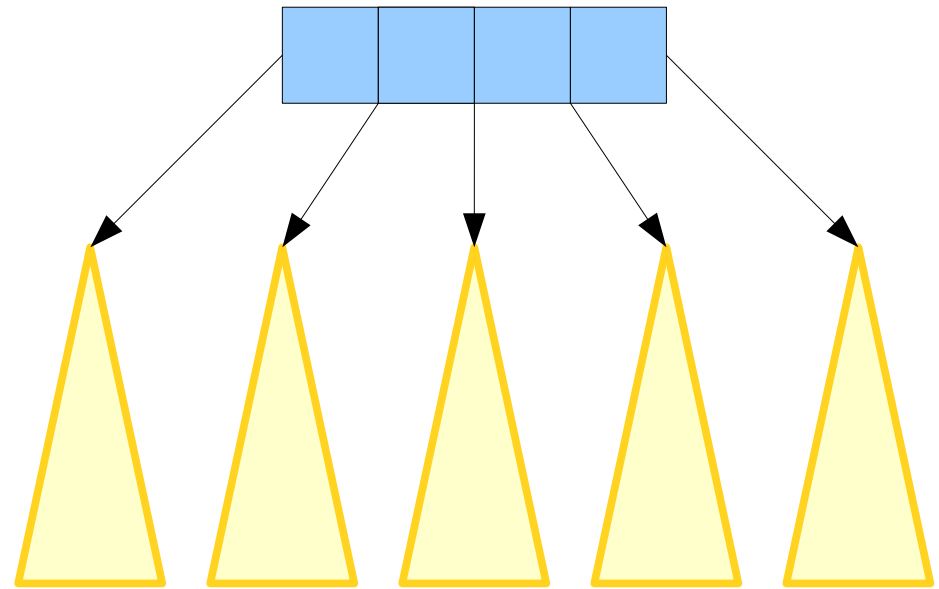
- **Option 2:** Use a binary search.
- Cost per node:  $O(\log b)$ .
- Nodes visited:  $O(\log_b n)$ .
- Total cost:  
 $O(\log b) \cdot O(\log_b n)$   
 $= O(\log b \cdot \log_b n)$   
 $= O(\log b \cdot (\log n) / (\log b))$   
 $= \mathbf{O(\log n)}$ .



That's the same as for binary search or a balanced BST.  
Why is that?

# Analyzing Efficiency

- Suppose we have a B-tree of order  $b$ .
- What is the worst-case runtime of inserting a key into the B-tree?
- Each insertion visits  $O(\log_b n)$  nodes, and in the worst case we have to split every node we see.
- **Answer:**  $O(b \log_b n)$ .



# Analyzing Efficiency

- The cost of an insertion in a B-tree of order  $b$  is  $O(b \log_b n)$ .
- What's the best choice of  $b$  to use here?
- Note that

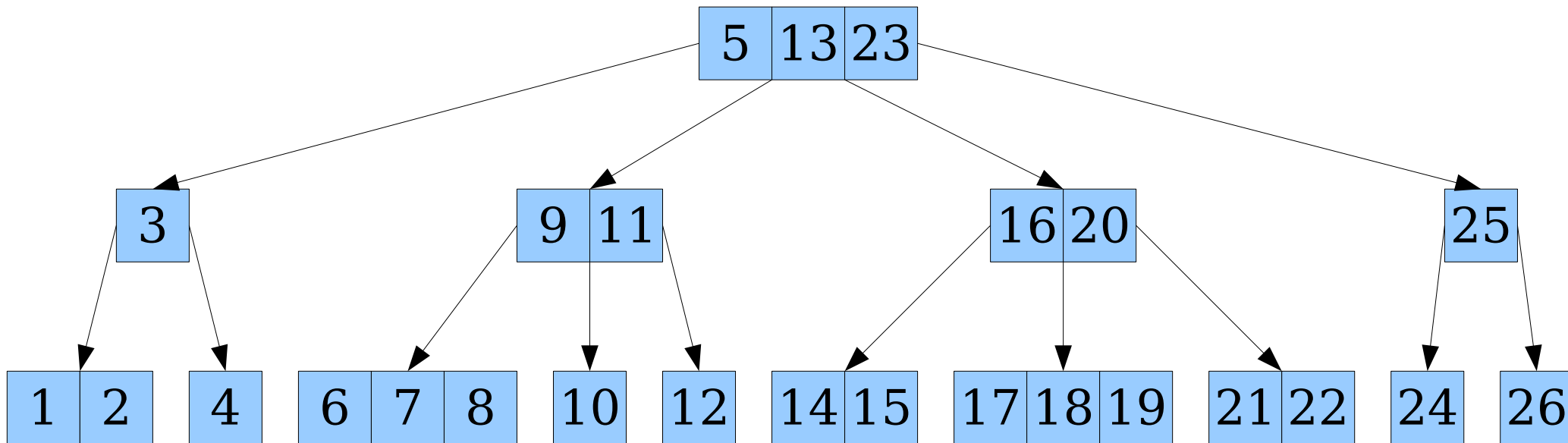
$$\begin{aligned} & b \log_b n \\ &= b (\log n / \log b) \\ &= (b / \log b) \log n. \end{aligned}$$

**Fun fact:** This is the same time bound you'd get if you used a  $b$ -ary heap instead of a binary heap for a priority queue.

- What choice of  $b$  minimizes  $b / \log b$ ?
- **Answer:** Pick  $b = e$ . (Or rather,  $b = \lfloor e \rfloor = 2$ .)

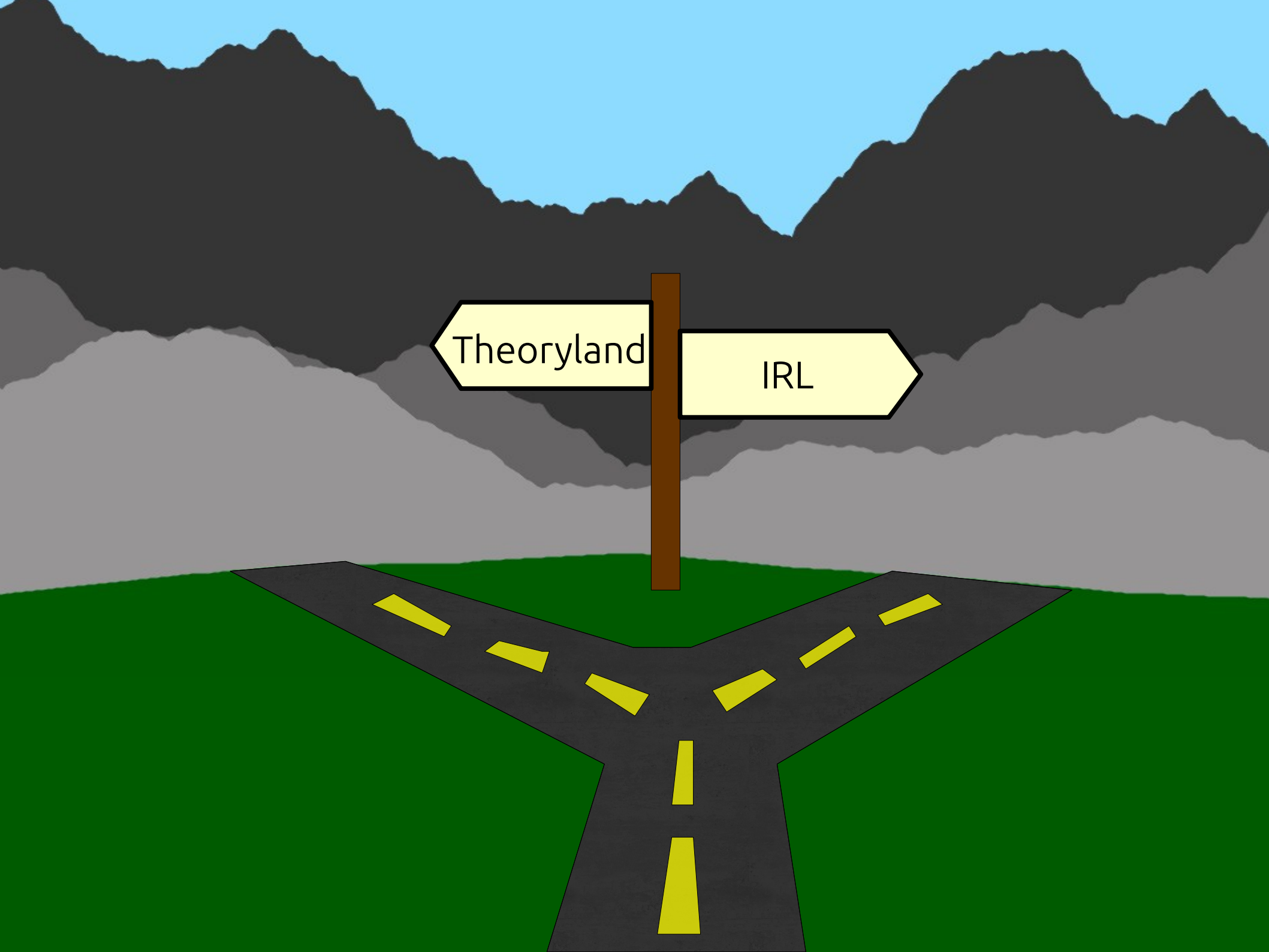
# 2-3-4 Trees

- A **2-3-4 tree** is a B-tree of order 2. Specifically:
  - each node has between 1 and 3 keys;
  - each node is either a leaf or has one more child than key; and
  - all leaves are at the same depth.
- You actually saw this B-tree earlier! It's the type of tree from our insertion example.



# The Story So Far

- A B-tree supports
  - lookups in time  $O(\log n)$ , and
  - insertions in time  $O(b \log_b n)$ .
- Picking  $b$  to be around 2 or 3 makes this optimal in Theoryland.
  - The 2-3-4 tree is great for that reason.
- ***Plot Twist:*** In practice, you most often see choices of  $b$  like 1,024 or 4,096.
- ***Question:*** Why would anyone do that?



Theoryland

IRL

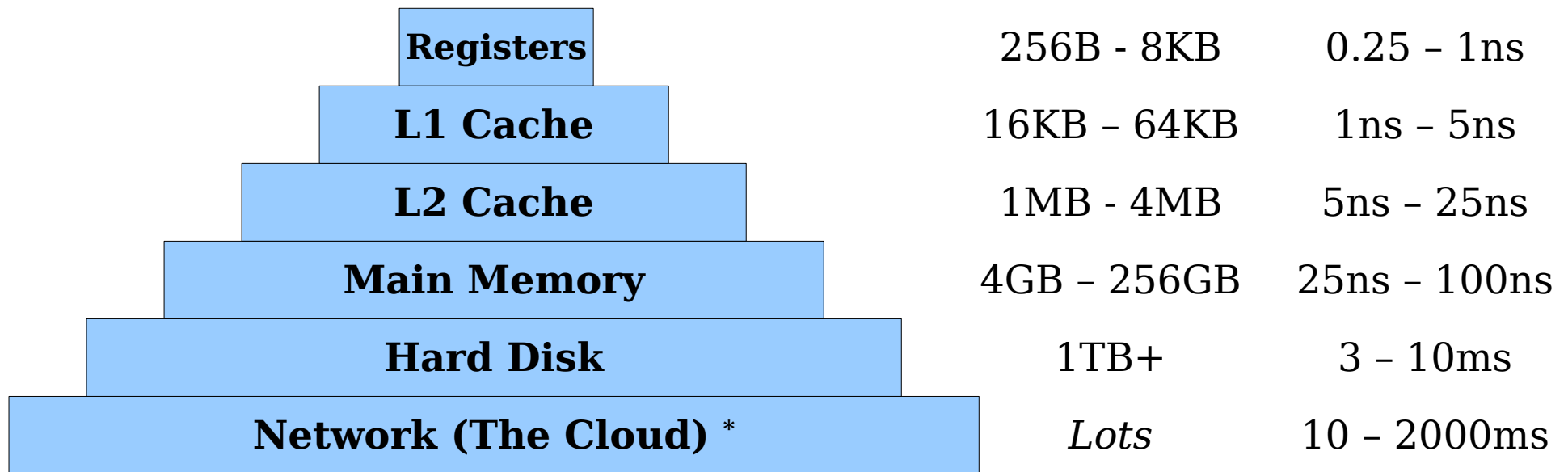
# The Memory Hierarchy

# Memory Tradeoffs

- There is an enormous tradeoff between *speed* and *size* in memory.
- SRAM (the stuff registers are made of) is fast but very expensive:
  - Can keep up with processor speeds in the GHz.
  - SRAM units can't be easily combined together; increasing sizes require better nanofabrication techniques (difficult, expensive).
- Hard disks are cheap but very slow:
  - As of 2025, you can buy a 4TB hard drive for about \$85.
  - As of 2025, good disk seek times for magnetic drives are measured in ms (about two to four million times slower than a processor cycle!)

# The Memory Hierarchy

- **Idea:** Try to get the best of all worlds by using multiple types of memory.



<b>Registers</b>	256B - 8KB	0.25 - 1ns
<b>L1 Cache</b>	16KB - 64KB	1ns - 5ns
<b>L2 Cache</b>	1MB - 4MB	5ns - 25ns
<b>Main Memory</b>	4GB - 256GB	25ns - 100ns
<b>Hard Disk</b>	1TB+	3 - 10ms
<b>Network (The Cloud) *</b>	<i>Lots</i>	10 - 2000ms

\* in some data centers, it's faster to store all data in RAM and access it over the network than to use magnetic disks!

# External Data Structures

- Suppose you have a data set that's *way* too big to fit in RAM.
- The data structure is on disk and read into RAM as needed.
- Data from disk doesn't come back one *byte* at a time, but rather one *page* at a time.
- **Goal:** Minimize the number of disk reads and writes, not the number of instructions executed.



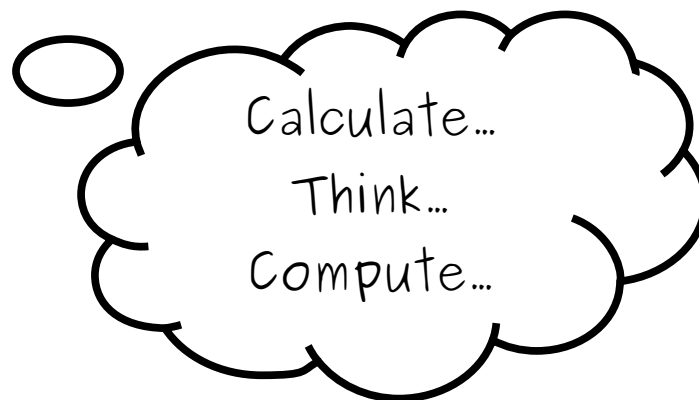
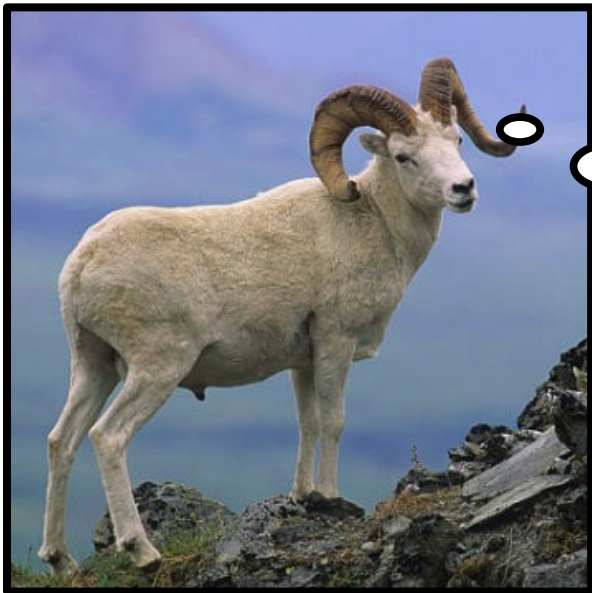
“Please give me 4KB  
starting at location *addr1*”



1101110010111011110001...

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# Analyzing B-Trees

- Suppose we tune  $b$  so that each node in the B-tree fits inside a single disk page.
- We *only* care about the number of disk pages read or written.
  - It's so much slower than RAM that it'll dominate the runtime.
- **Question:** What is the cost of a lookup in a B-tree in this model?
  - Answer: The height of the tree,  $O(\log_b n)$ .
- **Question:** What is the cost of inserting into a B-tree in this model?
  - Answer: The height of the tree,  $O(\log_b n)$ .

# External Data Structures

- Because B-trees have a huge branching factor, they're great for on-disk storage.
  - Disk block reads/writes are slow compared to CPU operations.
  - The high branching factor minimizes the number of blocks to read during a lookup.
  - Extra work scanning inside a block offset by these savings.
- Major use cases for B-trees and their variants (B<sup>+</sup>-trees, H-trees, etc.) include
  - databases (huge amount of data stored on disk);
  - file systems (ext4, NTFS, ReFS); and, recently,
  - in-memory data structures (due to cache effects).

# Analyzing B-Trees

- The cost model we use will change our overall analysis.
- Cost is number of operations:  
 **$O(\log n)$**  per lookup,  **$O(b \log_b n)$**  per insertion.
- Cost is number of blocks accessed:  
 **$O(\log_b n)$**  per lookup,  **$O(\log_b n)$**  per insertion.
- Going forward, we'll use operation counts as our cost model, though there's a ton of research done on designing data structures that are optimal from a cache miss perspective!

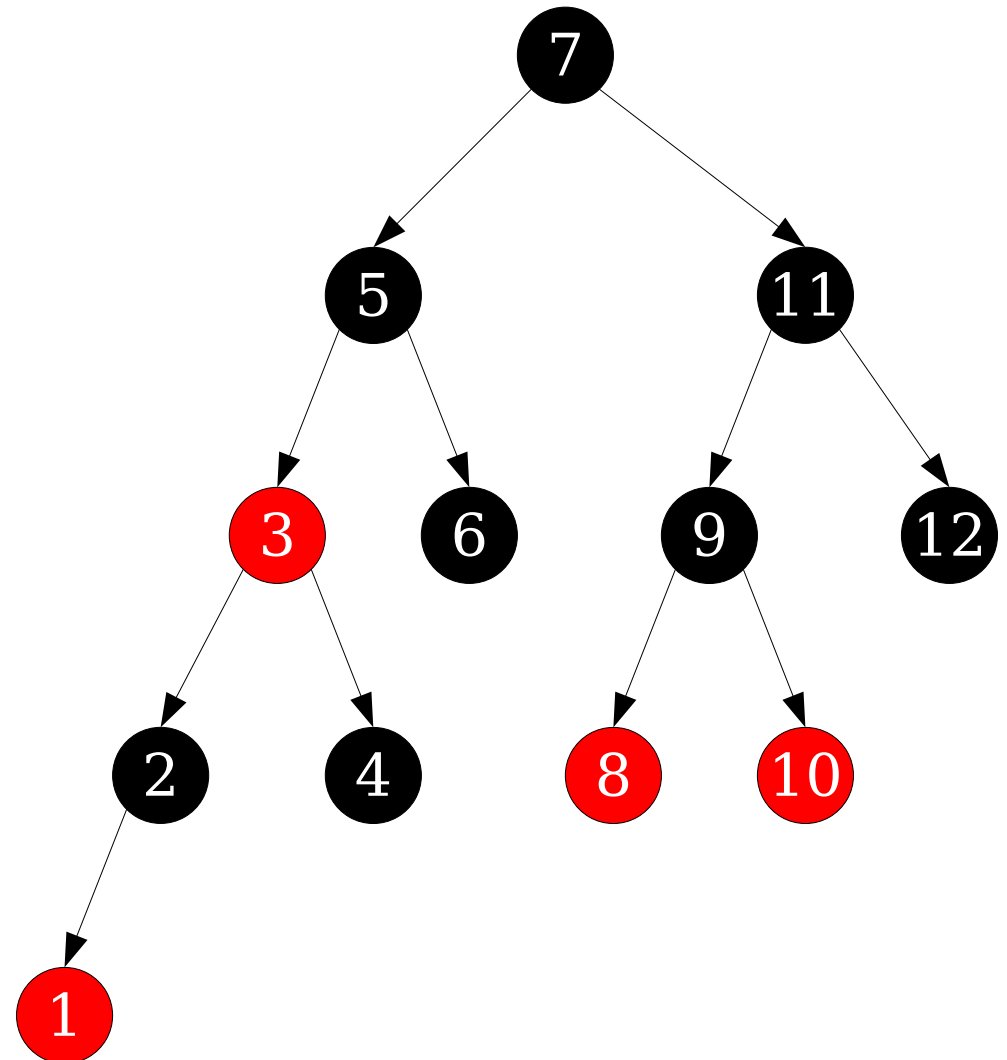
# The Story So Far

- We've just built a simple, elegant, balanced multiway tree structure.
- We can use them as balanced trees in main memory (2-3-4 trees).
- We can use them to store huge quantities of information on disk (B-trees).
- We've seen that different cost models are appropriate in different situations.

So... red/black trees?

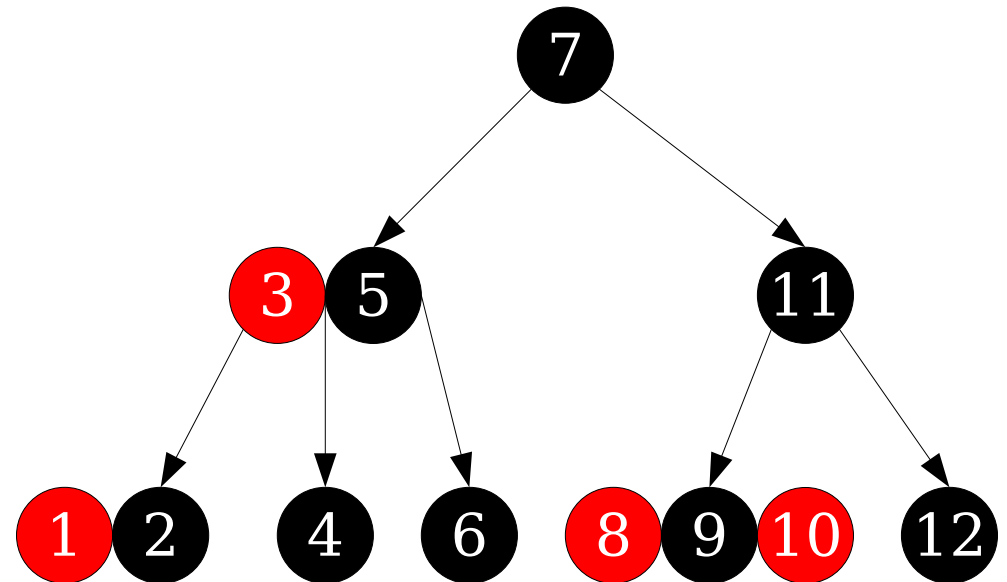
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# Red/Black Trees

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  - Every node is either red or black.
  - The root is black.
  - No red node has a red child.
  - Every root-null path in the tree passes through the same number of black nodes.
- After we hoist red nodes into their parents:
  - Each “meta node” has 1, 2, or 3 keys in it. (No red node has a red child.)
  - Each “meta node” is either a leaf or has one more child than key. (Root-null path property.)
  - Each “meta leaf” is at the same depth. (Root-null path property.)



***This is a  
2-3-4 tree!***

# Data Structure Isometries

- Red/black trees are an *isometry* of 2-3-4 trees; they represent the structure of 2-3-4 trees in a different way.
- Many data structures can be designed and analyzed in the same way.
- ***Huge advantage:*** Rather than memorizing a complex list of red/black tree rules, just think about what the equivalent operation on the corresponding 2-3-4 tree would be and simulate it with BST operations.

# Next Time

- ***Deriving Red/Black Trees***
  - Figuring out rules for red/black trees using our isometry.
- ***Tree Rotations***
  - A key operation on binary search trees.
- ***Augmented Trees***
  - Building data structures on top of balanced BSTs.